radius. This picture is consistent with the observed behavior of the Tokamak plasma radius. This matter of the need for diffusive development resulting in a vacuum magnetic buffer does not arise in the case of pinches, since the manner of their production automatically introduces such a buffer.

The failure of Tokamaks to exceed pressure balance with the poloidal magnetic field [i.e., Eq. (3)] is also consistent with the rigid drift model. This follows from the previously discussed fact that exact collisionless axisymmetric toroidal equilibria cannot depend on the toroidal magnetic field. Since this field can then only be a vacuum field, it cannot enter into the pressure balance relation.

The theoretical support for the existence of the postulated large anomalous resistivity follows from the argument that any equilibrium which is significantly different from a rigid drift equilibria necessarily possesses large pressure anisotropy and large shear in the drift velocity. In addition to those already known, there are undoubtedly many new high-frequency instabilities, resulting from pressure anisotropy and velocity shear which are associated with plasma turbulence and anomalous resistivity. These sources of instability vanish only when the plasma has evolved to a state sufficiently close to a rigid drift equilibrium.

With regard to low-frequency stability (in which the electrons can be approximately described by the fluid equation $\mathbf{E} + \mathbf{u}_e \times \mathbf{B} = 0$), it can be shown that the marginal stability boundary for the rigid drift model (for ions) is indentical to the magnetohydrodynamic marginal stability boundary. This follows because for the marginal state it turns out that the pressure is isotropic for both equilibrium and perturbation for the rigid drift model. The growth rates, of course, are much different.

The unique properties of the rigid drift model and the apparent fact that they are consistent with experimental observation, are the basis for proposing this model.

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Comments

COMMENTS refer to papers published in The Physics of Fluids *and are subject to the same length limitation as Research Notes. The Board of Editors will not hold itself responsible for the opinions expressed in the Comments.*

Comments on "Numerical Studies of Viscous Flow around Circular Cylinders"

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(Received 2 May 1969)

It is claimed by Hamielec and Raal[1] that their computations improve upon the extrapolation procedure of Keller and Takami[2] which is considered "inadequate" and "could presumably lead to appreciable errors." However, the authors clearly do not understand the procedure of Keller and Takami or else do not understand the nature of Imai's asymptotic solution, or both.

In the first place there is no reason (other than the assertion of Hamielec and Raal), for the computed solution to be independent of the radius r_∞ at which matching to the asymptotic solution occurs. In fact, it must depend upon r_∞ (unless the asymptotic solution were the exact solution) and this is the basis for the extrapolation $r_\infty \to \infty$, originated by Keller and Takami. Hamielec and Raal have simply imposed the free stream flow conditions at r_∞ and worse, required zero vorticity there and then used a modified (and perhaps less valid) form of the original $r_\infty \to \infty$ extrapolation. The fact that Keller and Takami use three values of r_∞ to extrapolate a quadratic fit in $1/r_\infty$ to zero means that they allow the drag to be of the form: $a_0 + a_1/r_\infty + a_2/r_\infty^2$. The "new extrapolation technique" of Hamielec and Raal simply assumes that $a_1 = 0$, which is clearly less accurate and not justified. The validity and accuracy of any extrapolation technique is a rather technical question in the theory of approximation whose discussion seems out of place here. Of course, there is no reason to expect that the simple polynomial form indicated above is best in any sense but an "improvement" should at least use more accuracy or a different extrapolation function.

The assumption by Hamielec and Raal that the computed drag should be a monotonic function of r_∞ is unwarranted. But to imply that the calculations of Keller and Takami are inaccurate since this *assumed* monotone behavior was not observed is irresponsible. In fact, the published[2] and about to be published[3] results of Keller and Takami reveal that their drag is monotone increasing in r_∞ for $\mathrm{Re} \leq 2$, is monotone decreasing in r_∞ for $\mathrm{Re} \geq 6$, and the switchover occurs at about $\mathrm{Re} = 4$. The convergence test employed by Hamielec and Raal is a very poor one. The relaxation scheme employed yields that

$$|\psi_n - \psi_{n-1}| = W\,|\psi_n^* - \psi_{n-1}|,$$

$$|\omega_n - \omega_{n-1}| = V\,|\omega_n^* - \omega_{n-1}|.$$

Unfortunately, the authors never tell how ψ_n^* and ω_n^* are computed. But since V is taken as small as $V = 0.02$ the test $|\omega_n - \omega_{n-1}| \leq 10^{-4}$ is not too difficult to satisfy if ω_n^* is computed in any reasonable way using ψ_{n-1} and ω_{n-1}. Finally they never indicate how or where the drag was computed. A very severe test on computations of this sort is to see how the drag computed on various circles about the body varies.

In brief, the computations of Hamielec and Raal seem to be quite crude and of doubtful accuracy. It seems apparent that they have merely modified the procedures of Keller and Takami in some important details. This *reduced* the effectiveness of the method. It happens to be the case for many of these steady viscous flow problems that almost any reasonable numerical procedure yields qualitatively reasonable looking results (if it does not blow up) and quantitatively reasonable results for various specific quantities.

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Reply to Comments by H. B. Keller

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It is claimed by Keller[1] that the computational procedures used by Hamielec and Raal[2] are mere modifications of procedures originated by Keller and Takami[3] and that these modifications "reduce the effectiveness of the method." Keller states[1] "Hamielec and Raal have simply imposed the free stream flow condition at r_∞ and worse, required zero vorticity there and then used a modified (and perhaps less valid) form of the original $r_\infty \to \infty$ extrapolation."

It is evident that our method of computation differs significantly from theirs in the boundary conditions used for stream function and vorticity at the finite radius r_∞. Unlike Keller and Takami's method, however, our computations *invariably* (i.e., in the entire N_{Re} range covered) yield form, friction, and total drag coefficients decreasing monotonically as the outer boundary r_∞ is extended. Thus, our method provides an upper bound to values for C_{DF}, C_{DP}, and C_D for an unbounded fluid.

Inspection will show that our extrapolation procedure, in which the condition

$$\frac{\partial F}{\partial(1/r_\infty)_{r_\infty = \infty}} = 0$$

(i.e., $a_1 = 0$) with $F = C_D$, C_{DF}, etc., is enforced provides a conservative upper bound to values of the extrapolated function for an unbounded fluid and reduces the possibility of "overcorrecting" for unbounded fluid.

A further illustration of the consistent way in which computed drag coefficients decrease, even for the exceptionally large values of the outer boundary used, is given in Table [.

In Fig. 1 we compare our results and the extrapolation to $1/r_\infty = 0$ with the curious results obtained and extrapolated by Keller and Takami for $N_{\mathrm{Re}} = 4$. The increase in drag coefficients with r_∞ at low Reynolds numbers, the decrease with r_∞ at higher N_{Re} and the erratic behavior obtained at $N_{\mathrm{Re}} = 4$ by Keller and Takami indicate that their method is incapable of providing consistently either upper or lower bounds.

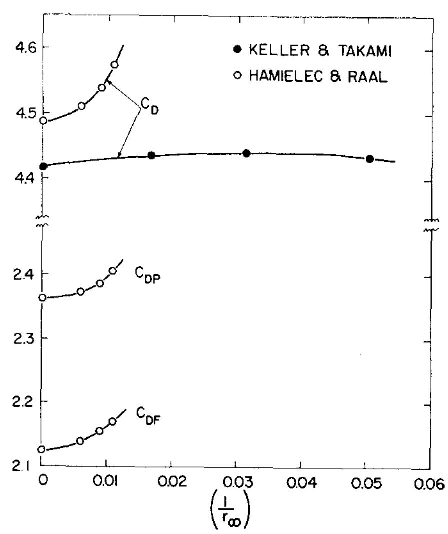

FIG. 1. Drag coefficients extrapolated for an unbounded fluid for $N_{\mathrm{Re}} = 4$.